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# Non-point integrable symmetries for equations on the lattice

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**Abstract.** We present a new class of non-point groups of transformations for scalar evolution chain equations. Then we construct the class of differential equations on the lattice which admits such group transformations.

## 1. Introduction

In the case of a continuous scalar evolution equation, the most general symmetry for which one can think of constructing the corresponding group transformation is given by a contact transformation. Equations of the form

$$u_t = f(t, x, u, u_x, u_{xx}, \ldots)$$

may admit a one-parametric group (1PG) of point and contact transformations of the form

$$\tilde{t} = \theta(\tau, t) \qquad \tilde{x} = \varphi(\tau, t, x, u, u_x) \qquad \tilde{u} = \psi(\tau, t, x, u, u_x).$$

The corresponding point and contact symmetries are of the form

$$u_\tau = g(t)u_t + h(t, x, u, u_x)$$

such that

$$u_{\tau t} = u_{t\tau}.$$

From the knowledge of such a symmetry, we can always reconstruct the group transformation by solving a system of coupled differential equations with boundary conditions.

In the case of differential equations on the lattice

$$u_{n,t} = f_n(t, u_n, u_{n\pm1}, u_{n\pm2}, \ldots) \tag{1}$$

the only class of symmetries which are known to be integrable, i.e. for which one can easily obtain the class of a 1PG of transformations, are the intrinsic point symmetries [3]

$$u_{n,\tau} = g(t)u_{n,t} + h_n(t, u_n) \tag{2}$$

whose corresponding group transformations read

$$\tilde{t} = \theta(\tau, t) \qquad \tilde{u}_n = \varphi_n(\tau, t, u_n). \tag{3}$$

Our aim is to show that equations (1) may admit a 1PG of transformations which is more general than those given by equation (3). In particular, in section 2 we show how any system of coupled evolution equations on the lattice can always be written as a scalar evolution equation on the lattice, possibly with $n$-dependent coefficients. Using this connection, we are able to present some nice examples of non-point transformations for scalar evolution equations on the lattice.

Starting from the general theory of 1PG transformations depending on few neighbouring points on the lattice, we are able to construct the simplest classes of integrable non-point symmetries.

In section 3 we present a few theorems which provide a class of evolution equations on the lattice depending on $u_{n+2}, u_{n+1}, \ldots, u_{n-2}$. Among those equations, there is the discrete Burgers equation

$$\dot{u}_n = u_n u_{n+1}(u_{n+2} - u_n) \tag{4}$$

and in section 4 we show how we can extend the class of solutions of this equation by the use of the so obtained non-point group transformations. Section 5 is devoted to a few concluding remarks.

## 2. Integrable non-point Lie symmetries

### 2.1. Existence of non-point non-trivial transformations

Let us introduce the transformation $T_M$ which allows us to rewrite a scalar chain as a system of $M$ equations and a system of $M$ chain equations as an $n$-dependent scalar chain. $T_M$ is given by

$$T_M : u_n \to U_k = (u_k^1, u_k^2, \ldots, u_k^M) \qquad \text{with} \quad u_k^i = u_{Mk+i}. \tag{5}$$

This transformation is obviously invertible. Such a transformation is well known, but it is used very seldom. Using the transformation (5), we can rewrite equation (1) as the system

$$U_{k,t} = F_k(t, U_k, U_{k\pm1}, \ldots). \tag{6}$$

To clarify this assertion let us consider a few examples. For instance, in the case of the Volterra equation

$$u_{n,t} = u_n(u_{n+1} - u_{n-1}) \tag{7}$$

we can have $M = 2$, as the equation involves three points on the lattice. The transformation (5) with $M = 2$ corresponds to splitting the points on the lattice into even and odd. Then equation (7) can be rewritten as the system

$$u_{k,t}^1 = u_k^1(u_k^2 - u_{k-1}^2) \qquad u_{k,t}^2 = u_k^2(u_{k+1}^1 - u_k^1).$$

On the other hand, the polynomial Toda chain

$$a_{k,t} = a_k(b_k - b_{k-1}) \qquad b_{k,t} = a_{k+1} - a_k$$

which is equivalent to equation (6) with $M = 2$, is expressed as an $n$-dependent scalar equation by interpreting $b_k$ and $a_k$ as the even and odd part of a function $u_n$

$$u_{n,t} = (p_n u_n + p_{n+1})(u_{n+1} - u_{n-1}) \tag{8}$$

where

$$p_n = 1 \quad \text{for odd } n \qquad \text{and} \qquad p_n = 0 \quad \text{for even } n. \tag{9}$$

Thus, in the field of discrete evolution equations, any system can be substituted by a scalar equation, i.e. one can think that there are no systems of equations and only scalar difference equations do exist. Moreover, by considering scalar lattice equations, one also obtains results on the theory for systems (6).

The same transformation also applies to the symmetries, local conservation laws (see, e.g., [4]) and 1PG of transformations. In particular, using the transformations $T_M$, we can obtain transformations for scalar equations (1) which are not point ones. Let $M = 2$, and the 1PG of point transformations be given by

$$\tilde{u}_k^1 = u_k^1 + \tau u_k^2 \qquad \tilde{u}_k^2 = u_k^2.$$

Using equation (5), we obtain the explicit formula

$$\tilde{u}_{2k+1} = \tilde{u}_k^1 = u_k^1 + \tau u_k^2 = u_{2k+1} + \tau u_{2k+2}$$
$$\tilde{u}_{2k+2} = \tilde{u}_k^2 = u_k^2 = u_{2k+2}.$$

Introducing $p_n$ according to equation (9), we have for all $n$

$$\tilde{u}_n = u_n + \tau p_n u_{n+1} \tag{10}$$

i.e. a scalar non-point 1PG of transformations.

As is well known, systems may admit non-point transformations when written down in triangular form (see, e.g., [1, 5, 6]). 1PGs of triangular transformations are well known, but are not very well investigated. A transformation for a system of equations is said to be triangular when, if one writes it down in matrix form, it is represented by a triangular matrix. In such a case the off-diagonal terms can depend on higher derivatives and one can still obtain the 1PG of transformations from the symmetries. As an example, in the case of systems of continuous equations, the following formulae:

$$\tilde{u} = u \qquad \tilde{v} = v + \tau u_x$$

define a 1PG of triangular transformations which are non-point ones. In the case of system (6) with $M = 2$, we define a triangular 1PG as

$$\tilde{u}_k^1 = u_k^1 \qquad \tilde{u}_k^2 = u_k^2 + \tau\varphi(u_{k+1}^1, u_k^1)$$

where $\varphi$ is an arbitrary function of two variables. For the corresponding $n$-dependent scalar transformation we obtain

$$\tilde{u}_n = u_n + \tau p_{n+1}\varphi(u_{n+1}, u_{n-1}) \tag{11}$$

where $p_n$ is defined in equation (9).

We can consider generalizations of equations (10) and (11) when no periodic function is involved. Consider, for example, a 1PG of transformations given by

$$\tilde{u}_n = u_n + \tau\lambda_n\varphi_n(t, u_{n+1}, u_{n-1}) \tag{12}$$

where $\varphi_n$ is an arbitrary $n$-dependent function of three variables, and $\lambda_n$ is an $n$-dependent constant satisfying the condition

$$\lambda_{n+1}\lambda_n = 0 \qquad \forall\, n. \tag{13}$$

The condition only means that if $\lambda_i \neq 0$ for some $n = i$, then $\lambda_{i+1} = \lambda_{i-1} = 0$, while if $\lambda_i = 0$ than $\lambda_{i+1}$ and $\lambda_{i-1}$ may be different from zero. The corresponding symmetry has the form

$$u_{n,\tau} = \lambda_n\varphi_n(t, u_{n+1}, u_{n-1}). \tag{14}$$

If for a given equation, one has a symmetry of the form (14), the corresponding group transformation is given by equation (12).

*2.2. Construction of transformations depending on just one shifted variable*

In this subsection we consider a class of Lie group transformations of the form

$$\tilde{u}_n = H_n(\tau, t, u_n, u_{n+1}) \qquad H_n(0, t, u_n, u_{n+1}) = u_n \tag{15}$$

$$\frac{\partial H_n}{\partial u_{n+1}} \neq 0 \qquad \text{for at least some } n \tag{16}$$

with $H_n$ an analytic function of all its arguments (not only of $\tau$). By imposing the group closure condition, it can be shown that such group transformations can be written in the form

$$\tilde{u}_n = \lambda_n \mathcal{R}_n(\tau, t, u_n, u_{n+1}) + (1 - \lambda_n)\mathcal{Q}_n(\tau, t, u_n) \tag{17}$$

$$\frac{\partial \mathcal{R}_n}{\partial u_{n+1}} \neq 0 \qquad \forall\, n \tag{18}$$

$$\lambda_n^2 = \lambda_n \quad \text{and} \quad \lambda_n \lambda_{n+1} = 0 \quad \forall n \qquad \lambda_n \neq 0 \quad \text{for some } n. \tag{19}$$

Here $\mathcal{R}_n$ and $\mathcal{Q}_n$ are such that

$$\mathcal{R}_n(0, t, u_n, u_{n+1}) = \mathcal{Q}_n(0, t, u_n) = u_n \tag{20}$$

with

$$\mathcal{Q}_n(\tau + \eta, t, u_n) = \mathcal{Q}_n(\eta, t, \mathcal{Q}_n(\tau, t, u_n)) \tag{21}$$

$$\mathcal{R}_n(\tau + \eta, t, u_n, u_{n+1}) = \mathcal{R}_n(\eta, t, \mathcal{R}_n(\tau, t, u_n, u_{n+1}), \mathcal{Q}_{n+1}(\tau, t, u_{n+1})). \tag{22}$$

As equation (15) represents a 1PG of Lie transformations, which, by definition, depend analytically on the group parameter $\tau$, $\mathcal{R}_n$ and $\mathcal{Q}_n$ can be represented by a Taylor expansion in $\tau$:

$$\mathcal{R}_n(\tau, t, u_n, u_{n+1}) = u_n + \tau r_n(t, u_n, u_{n+1}) + \tau^2 r_n^1(t, u_n, u_{n+1}) + \cdots \tag{23}$$

$$\mathcal{Q}_n(\tau, t, u_n) = u_n + \tau q_n(t, u_n) + \tau^2 q_n^1(t, u_n) + \cdots. \tag{24}$$

Introducing equations (23), (24) into (22), (21), we obtain two equations polynomial in $\tau$ and $\eta$ which must be satisfied identically for any $\tau$ and $\eta$ and define in a unique way the coefficients of (23) and (24). We obtain, for example,

$$q_n^1 = \frac{1}{2} q_n \frac{\partial q_n}{\partial u_n} \qquad q_n^2 = \frac{1}{3} q_n \frac{\partial q_n^1}{\partial u_n} \quad \ldots \tag{25}$$

$$r_n^1 = \frac{1}{2}\left( r_n \frac{\partial r_n}{\partial u_n} + q_{n+1} \frac{\partial r_n}{\partial u_{n+1}} \right) \qquad r_n^2 = \frac{1}{3}\left( r_n \frac{\partial r_n^1}{\partial u_n} + q_{n+1} \frac{\partial r_n^1}{\partial u_{n+1}} \right) \quad \ldots. \tag{26}$$

It is easy to show that by introducing the infinitesimal generator

$$\hat{X} = [\lambda_n r_n(t, u_n, u_{n+1}) + (1 - \lambda_n) q_n(t, u_n)]\partial_{u_n} \tag{27}$$

and its prolongation

$$pr\hat{X} = \hat{X} + [\lambda_{n+1} r_{n+1}(t, u_{n+1}, u_{n+2}) + (1 - \lambda_{n+1}) q_{n+1}(t, u_{n+1})]\partial_{u_{n+1}} \tag{28}$$

we obtain

$$\lambda_n \mathcal{R}_n(\tau, t, u_n, u_{n+1}) = \mathrm{e}^{\tau pr\hat{X}} \lambda_n u_n \tag{29}$$

$$(1 - \lambda_n)\mathcal{Q}_n(\tau, t, u_n) = \mathrm{e}^{\tau \hat{X}} (1 - \lambda_n) u_n. \tag{30}$$

The symmetries corresponding to equation (15) are given by

$$u_{n,\tau} = \lambda_n r_n(t, u_n, u_{n+1}) + (1 - \lambda_n) q_n(t, u_n) = S_n \tag{31}$$

$$\frac{\partial r_n}{\partial u_{n+1}} \neq 0 \qquad \forall\, n \tag{32}$$

where $\lambda_n$ is given by (19). Equations (31) and (32) are a straightforward consequence of the group transformation (17)–(19), and given (31) and (32), (17)–(19) are reobtained by taking into account (29) and (30). This form of the symmetries will be used to solve the classification problem and obtain the class of differential difference equations which possess this class of symmetries.

## 3. Equations with non-point integrable symmetries

Let us consider the following class of equations:

$$u_{n,t} = f_n(t, u_{n+M}, u_{n+M-1}, \ldots, u_{n+N}) \tag{33}$$

$$N \leqslant M \qquad M \geqslant 1 \qquad \frac{\partial f_n}{\partial u_{n+M}} \neq 0 \quad \forall n \tag{34}$$

and symmetries given by equations (31) and (32).

The compatibility condition for (31) and (33) takes the form

$$\frac{\partial S_n}{\partial t} + \frac{\partial S_n}{\partial u_n} f_n + \frac{\partial S_n}{\partial u_{n+1}} f_{n+1} = \sum_{k=N}^{M} \frac{\partial f_n}{\partial u_{n+k}} S_{n+k}. \tag{35}$$

Differentiating (35) with respect to $u_{n+M+1}$, we obtain

$$\lambda_n \frac{\partial r_n}{\partial u_{n+1}} \frac{\partial f_{n+1}}{\partial u_{n+M+1}} = \lambda_{n+M} \frac{\partial f_n}{\partial u_{n+M}} \frac{\partial r_{n+M}}{\partial u_{n+M+1}}. \tag{36}$$

No function can vanish here, but $\lambda_n$ (see (32) and (34)), and the values of $\lambda_n$ belong to the set $\{0, 1\}$. Hence

$$\lambda_{n+M} = \lambda_n \qquad \forall\, n. \tag{37}$$

Thus $\lambda_n$ must be *M-periodic*. In such a case, the symmetry (31) can be rewritten as a system of point symmetries for a system of $M$ equations. In the case $M = 1$, the condition (37) means that $\lambda_n$ is a constant. However, this is impossible (see equation (19)). So, if $M = 1$, equation (33) cannot possess symmetries of the form (31). This is the case of the Toda and Volterra lattices (7) and (8) and many other integrable lattice equations which are of the form (33) and (34) with $M = 1$ and $N = -1$. The Burgers-type equation

$$u_{n,t} = u_n(u_{n+1} - u_n) \tag{38}$$

belongs to the class (33) and (34) with $M = 1$ and $N = 0$; so it also does not have symmetries of the form (31).

We will consider here the case $M = 2$, restricting ourselves to the case of $n$-independent equations for the sake of simplicity. This class contains, for instance, the simplest higher equations of the Volterra and Burgers hierarchies, (7) and (38). More precisely, we will consider the class of equations

$$u_{n,t} = f(t, u_{n+2}, u_{n+1}, \ldots, u_{n+N}) \tag{39}$$

$$N \leqslant 0 \qquad \frac{\partial f}{\partial u_{n+2}} \neq 0. \tag{40}$$

Condition (37), for $M = 2$, means that $\lambda_n$ is nothing but the two periodic projector $p_n$ defined as in equation (9). So we will be considering symmetries of the form

$$u_{n,\tau} = p_n r_n(t, u_n, u_{n+1}) + q_n(t, u_n) = S_n \tag{41}$$

$$\frac{\partial r_n}{\partial u_{n+1}} \neq 0 \qquad \forall\, n. \tag{42}$$

Under such general assumptions, we provide below two results (expressed as propositions) in which the form of symmetries is almost completely defined (up to functions depending only on $n$ and $t$). After that the classification for any given $N$ will be just technical work which is carried out by using a symbolic manipulation program.

We will carry out the classification up to point transformations of the form

$$\tilde{t} = \theta(t) \qquad \tilde{u}_n(\tilde{t}) = \phi(t, u_n(t)) \tag{43}$$

which preserve the form of equations (39) and (40).

Let us rewrite condition (36) in terms of

$$R_n(t, u_n, u_{n+1}) = \frac{\partial r_n}{\partial u_{n+1}} \qquad F(t, u_{n+2}, \ldots, u_{n+N}) = \frac{\partial f}{\partial u_{n+2}}. \tag{44}$$

Then it takes the form

$$p_n R_n(DF) = p_n F R_{n+2} \tag{45}$$

where $D$ is the shift operator such that

$$DF(t, u_{n+2}, \ldots, u_{n+N}) = F(t, u_{n+3}, \ldots, u_{n+N+1}). \tag{46}$$

We can then state the following proposition.

**Proposition 1.** *Condition (45) is satisfied if and only if an equation of the class (39) and (40), and its symmetry of the class (41) and (42) can be expressed (up to point transformations (43)) in the form*

$$u_{n,t} = \frac{\partial r}{\partial u_{n+1}}(Dr) + g(t, u_{n+1}, \ldots, u_{n+N}) \tag{47}$$

$$u_{n,\tau} = p_n r + q_n(t, u_n) \tag{48}$$

*where $r$ is an $n$-independent function*

$$r = r(t, u_n, u_{n+1}) \qquad \frac{\partial r}{\partial u_{n+1}} \neq 0. \tag{49}$$

**Proof.** If

$$\frac{\partial F}{\partial u_{n+N}} \neq 0 \qquad N \leqslant -1$$

then equation (45) implies $p_n \frac{\partial F}{\partial u_{n+N}} R_{n+2} = 0$ for any $n$, i.e. $F$ does not depend on $u_{n+N}$. This contradiction shows that

$$F = F(t, u_{n+2}, u_{n+1}, u_n).$$

Now we are going to show that there exist functions $R$, $Q$ such that

$$R_n = R(t, u_n, u_{n+1}) \neq 0 \quad \forall n \qquad F = QR(DR) \qquad Q = Q(t) \neq 0. \tag{50}$$

Condition (45) is equivalent to

$$p_n \frac{(DF)}{R_{n+2}} = p_n \frac{F}{R_n} = p_n K_{n+1} \tag{51}$$

where the function $K_n$, defined by equation (51), depends only on $t$, $u_n$ and $u_{n+1}$. We can rewrite equation (45) as

$$p_n F = p_n R_n K_{n+1} \qquad p_{n+1} F = p_{n+1} K_n R_{n+1}. \tag{52}$$

Let us denote $R_n$ at a fixed number $n = i$ (where $i$ is an odd number) by $R$:

$$R(t, x, y) = R_i(t, x, y) \tag{53}$$

where $x = u_i$ and $y = u_{i+1}$. Putting, at first, $n = i$ in the first of equations (52), we obtain (see equation (9))

$$F(t, z, y, x) = R(t, x, y) K_{i+1}(t, y, z) \tag{54}$$

where $z = u_{i+2}$, and then $n = i - 1$ in the second of equations (52) we obtain

$$F(t, z, y, x) = K_{i-1}(t, x, y) R(t, y, z) \tag{55}$$

where now $z = u_{i+1}$, $y = u_i$ and $x = u_{i-1}$. This implies

$$\frac{K_{i+1}(t, y, z)}{R(t, y, z)} = \frac{K_{i-1}(t, x, y)}{R(t, x, y)} = Q(t, y)$$

i.e.

$$K_{i-1}(t, x, y) = R(t, x, y) Q(t, y) \tag{56}$$

and thus for $F$ (see equation (55)) we obtain

$$F(t, z, y, x) = Q(t, y) R(t, x, y) R(t, y, z). \tag{57}$$

We see that we must have $Q \neq 0$ and $R \neq 0$ as $F \neq 0$.

Following the same procedure which we used to obtain equations (54) and (53), the first of equations (52) for $n = i - 2$ gives (see also equation (56))

$$F(t, z, y, x) = R_{i-2}(t, x, y) K_{i-1}(t, y, z) = R_{i-2}(t, x, y) R(t, y, z) Q(t, z). \tag{58}$$

Comparing equation (58) with (57), we obtain

$$Q(t, y) R(t, x, y) = R_{i-2}(t, x, y) Q(t, z).$$

This means $Q = Q(t)$, as $R_n$ does not vanish for any $n$. Taking into account equation (57), we are led to the following representation:

$$F(t, u_{n+2}, u_{n+1}, u_n) = Q(t) R(t, u_n, u_{n+1}) R(t, u_{n+1}, u_{n+2})$$

which proves the second part of equation (50).

If we return to equation (51) and use the definition of $F$ given in the second part of equation (50), we have

$$p_n \frac{Q(DR)(D^2 R)}{R_{n+2}} = p_n \frac{QR(DR)}{R_n}. \tag{59}$$

Introducing the quantity

$$G_n = \frac{R(t, u_n, u_{n+1})}{R_n(t, u_n, u_{n+1})}$$

equation (59) can be rewritten as

$$p_n G_{n+2} = p_n G_n. \tag{60}$$

This means that for $n$ odd the functions $G_n$ are all equal (see definition (9) of $p_n$) and, therefore, are equal to $G_i$. It follows from equation (53) that $G_i = 1$, hence

$$R_n(t, u_n, u_{n+1}) = R(t, u_n, u_{n+1}) \tag{61}$$

for all $n$ odd. Only functions $r_n$ with $n$ odd play a role in equation (41) for symmetries; so we may define the functions $r_n$ for the other values of $n$ (and therefore $R_n$) as we like. So, we consider equation (61) to be valid for all integer $n$ thus proving equation (50).

Consequently, we can introduce a function $r$ such that

$$r = r(t, u_n, u_{n+1}) \qquad \frac{\partial r}{\partial u_{n+1}} = R \neq 0. \tag{62}$$

Then we have (see equations (44) and (50))

$$\frac{\partial r_n}{\partial u_{n+1}} = \frac{\partial r}{\partial u_{n+1}} \qquad \frac{\partial f}{\partial u_{n+2}} = Q(t)\frac{\partial r}{\partial u_{n+1}}\left(D\frac{\partial r}{\partial u_{n+1}}\right). \tag{63}$$

Integrating equation (63), we are led to the following representation for our equation and its symmetry:

$$u_{n,t} = Q(t)\frac{\partial r}{\partial u_{n+1}}(Dr) + g(t, u_{n+1}, \ldots, u_{n+N}) \qquad u_{n,\tau} = p_n r + q_n(t, u_n).$$

We can apply a point transformation (43) with $\theta' = Q$, $\phi = u_n(t)$ and redefine the functions $r, g, q_n$. After that we obtain equations (47)–(49). $\square$

Let us consider equations of the form of equation (47) and symmetries of the form of equations (48) and (49). Differentiating the compatibility condition (35) with $M = 2$ and $f_n = f$ (where $f$ and $S_n$ are given by the right-hand side of equations (47) and (48)) with respect to $u_{n+2}$ and dividing by $F$ (see equation (44)), we are led to a second condition which can be written in the form

$$(\log F)_\tau = (D - 1)\left(\frac{1}{F}\frac{\partial S_{n+1}}{\partial u_{n+2}}\frac{\partial f}{\partial u_{n+1}} - \frac{\partial S_{n+1}}{\partial u_{n+1}} - \frac{\partial S_n}{\partial u_n}\right). \tag{64}$$

Equation (64) has the form of a local conservation law, and we can express it as the condition that $(\log F)_\tau \sim 0$, i.e. that there exists a function $\Omega_n$ (depending only on a finite number of variables $u_{n+k}$), such that $(\log F)_\tau = \Omega_{n+1} - \Omega_n$. Using the representation of $F$ given in equation (50) with $Q = 1$ and $R$ defined by equation (62) and its consequences

$$(\log F)_\tau = (D + 1)(\log R)_\tau = 2(\log R)_\tau + (D - 1)(\log R)_\tau$$

we find that

$$(\log R)_\tau \sim 0. \tag{65}$$

This condition means that $\log R$ is a conserved density of equation (48). This is a condition only for the symmetry (48), and it is a strong restriction. This allows us to write the following proposition.

**Proposition 2.** *Equations (47)–(49) satisfy condition (64) if and only if they can be written (up to point transformations (43)) in one of the following 2-forms.*

*In the linear case:*

$$u_{n,t} = u_{n+2} + b(t)u_{n+1} + h(t, u_n, \dots, u_{n+N}) \tag{66}$$

$$u_{n,\tau} = p_n\beta(t)u_{n+1} + (\gamma(t) + \mu_n(t))u_n + \nu_n(t) \tag{67}$$

$$\beta \neq 0 \qquad \mu_{n+1} + \mu_n = p_{n+1}\beta b \tag{68}$$

*and in the nonlinear case:*

$$u_{n,t} = u_n u_{n+1} u_{n+2} - u_n^2 u_{n+1} + b(t)u_n u_{n+1} + h(t, u_n, \dots, u_{n+N}) \tag{69}$$

$$u_{n,\tau} = p_n u_n u_{n+1} - p_{n+1}u_n^2 + \mu_n(t)u_n \tag{70}$$

$$\mu_{n+1} + \mu_n = (p_{n+1} - p_n)b. \tag{71}$$

*Here $N \leqslant 0$, and $p_n$ is defined according to equation (9).*

**Proof.** From equation (65) we obtain

$$\frac{\partial \log R}{\partial u_n} S_n + \frac{\partial \log R}{\partial u_{n+1}} S_{n+1} = \Omega_{n+1} - \Omega_n \tag{72}$$

where $\Omega_n$ may depend only on $t, u_n, u_{n+1}$. The second derivative of equation (72) with respect to $u_n$ and $u_{n+2}$ gives

$$\frac{\partial^2 \log R}{\partial u_n \partial u_{n+1}} \frac{\partial S_{n+1}}{\partial u_{n+2}} = 0. \tag{73}$$

As $\frac{\partial S_{n+1}}{\partial u_{n+2}} = p_{n+1}(DR)$, from equation (73) it follows that $\frac{\partial^2 \log R}{\partial u_n \partial u_{n+1}} = 0$. So $R$ can be written as $R = a(t, u_n)\tilde{c}(t, u_{n+1})$. Then for the function $r$, we obtain

$$r = a(t, u_n)c(t, u_{n+1}) + \tilde{r}(t, u_n)$$

where $a\frac{\partial c}{\partial u_{n+1}} \neq 0$ due to equation (49). The function $\tilde{r}$ disappears if we redefine $g$ and $q_n$ in (47) and (48). Applying the point transformation $\tilde{u}_n = c(t, u_n)$ and redefining $a$, $g$ and $q_n$, we obtain the same formula for $r$, but with $c = u_{n+1}$ and $\tilde{r} = 0$. So, we are led to equations (47) and (48) with

$$r = a(t, u_n)u_{n+1} \qquad (a \neq 0) \tag{74}$$

instead of (49).

As $R = a(t, u_n)$, from equation (65) we have

$$(\log a)_\tau = p_n \frac{\partial a}{\partial u_n} u_{n+1} + q_n \frac{\partial \log a}{\partial u_n} = \Omega_{n+1} - \Omega_n \tag{75}$$

where $\Omega_n = \Omega_n(t, u_n)$. Applying the operator $\frac{\partial^2}{\partial u_n \partial u_{n+1}}$ to equation (75), we obtain $p_n \frac{\partial^2 a}{\partial u_n^2} = 0$, which implies

$$a = \alpha(t)u_n + \beta(t). \tag{76}$$

Now we consider separately the two different cases $\alpha = 0$ and $\alpha \neq 0$.

In the case $\alpha = 0$, after obvious transformations, we are led, taking into account equations (47), (48), (74) and (76), to the following form of the equation and symmetry:

$$u_{n,t} = u_{n+2} + g(t, u_{n+1}, \dots, u_{n+N}) \tag{77}$$

$$u_{n,\tau} = p_n\beta(t)u_{n+1} + q_n(t, u_n) \qquad \beta \neq 0. \tag{78}$$

Introducing the functions

$$G(t, u_{n+1}, \dots, u_{n+N}) = \frac{\partial g}{\partial u_{n+1}} \qquad Q_n(t, u_n) = \frac{\partial q_n}{\partial u_n}$$

the condition (64) gives

$$(D-1)(p_{n+1}\beta G - Q_{n+1} - Q_n) = 0$$

and hence

$$p_{n+1}\beta G = Q_{n+1} + Q_n - 2\gamma(t) \tag{79}$$

where $\gamma$ is an arbitrary function, and $G = G(t, u_{n+1}, u_n)$.

Moreover, $G$ must be of the form $G = DA(t, u_n) + B(t, u_n)$. Introducing this into equation (79), we have

$$Q_{n+1} - \gamma - p_{n+1}\beta(DA) = -Q_n + \gamma + p_{n+1}\beta B = \eta_n(t)$$

or else

$$Q_n - \gamma = p_n\beta A + \eta_{n-1} = p_{n+1}\beta B - \eta_n.$$

If we consider the second equality for odd and even values of $n$, we see that $A$ and $B$ do not depend on $u_n$, then $Q_n$ also does not depend. So, $G$ is just a function of $t$, and $q_n$ is a linear function of $u_n$:

$$G = b(t) \qquad q_n = (\gamma(t) + \mu_n(t))u_n + \nu_n(t). \tag{80}$$

Introducing these formulae in equation (79), we obtain the second of conditions (68). Taking into account equations (77), (78) and (80), we are led to equations (66)–(68), i.e. we have proved the first part of proposition 2.

Let us now consider the second case $\alpha \neq 0$ in equation (76). Applying the transformation $\tilde{u}_n = \alpha u_n + \beta$ to equations (47), (48), (74) and (76), we obtain the same formulae as before, but with

$$r = u_n u_{n+1}. \tag{81}$$

This means that our equation and its symmetries now have the form

$$u_{n,t} = f = u_n u_{n+1} u_{n+2} + g(t, u_{n+1}, \dots, u_{n+N}) \tag{82}$$

$$u_{n,\tau} = S_n = p_n u_n u_{n+1} + q_n(t, u_n). \tag{83}$$

From condition (65) we obtain

$$(\log u_n)_\tau = p_n u_{n+1} + \frac{q_n}{u_n} \sim p_{n+1}u_n + \frac{q_n}{u_n} = \Omega_{n+1} - \Omega_n$$

where $\Omega_n$ is just a $t$-dependent function. So, $p_{n+1}u_n + q_n/u_n$ is a function $\mu_n(t)$ and thus

$$q_n = -p_{n+1}u_n^2 + \mu_n(t)u_n. \tag{84}$$

Let us take into account condition (64), which we can now rewrite as

$$(D-1)\frac{p_{n+1}}{u_n}\frac{\partial g}{\partial u_{n+1}} = p_{n+1}u_n - p_n u_{n+1} + \mu_{n+2} + \mu_{n+1}. \tag{85}$$

Introducing $\nu_n$, such that

$$\frac{p_{n+1}}{u_n}\frac{\partial g}{\partial u_{n+1}} = -p_{n+1}u_n + \nu_n \tag{86}$$

we obtain from equation (85) the relation

$$\nu_{n+1} - \nu_n = \mu_{n+2} + \mu_{n+1}. \tag{87}$$

Condition (87) implies that $\nu_n = \nu_n(t)$. Considering equation (86) for $n$ even (see the definition of $p_n$), we obtain for $g$ an equation of the form

$$\frac{1}{y}\frac{\partial g(t, x, y, \ldots)}{\partial x} = -y + b(t)$$

which implies

$$g = -u_n^2 u_{n+1} + b(t) u_n u_{n+1} + h(t, u_n, \ldots, u_{n+N}). \tag{88}$$

Condition (86) is equivalent to

$$\nu_n = p_{n+1} b$$

and condition (87) is equivalent to equation (71) for $\mu_n$. Taking into account equations (84) and (88), we obtain from equations (82) and (83), formulae (69)–(71) which conclude the proof of proposition 2. □

Starting from the results of proposition 2 we can easily prove by straightforward calculation the following theorem.

**Theorem 1.** *An equation of the form*

$$u_{n,t} = f(t, u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}) \qquad \frac{\partial f}{\partial u_{n+2}} \neq 0 \tag{89}$$

*possesses a symmetry of the form (41) and (42) if and only if that equation and its symmetry can be expressed (up to a point transformation (43) and a scaling $\tilde{\tau} = c\tau$) in one of the following two forms.*

*In the nonlinear case:*

$$u_{n,t} = u_n u_{n+1}(u_{n+2} - u_n) + \frac{h(t)}{u_{n-1}}\left(1 - \frac{u_n}{u_{n-2}}\right) \tag{90}$$

$$u_{n,\tau} = p_n u_n u_{n+1} - p_{n+1} u_n^2 + \delta(-1)^n u_n \tag{91}$$

*and in the linear case:*

$$u_{n,t} = u_{n+2} + h(t) u_{n-2} \tag{92}$$

$$u_{n,\tau} = p_n u_{n+1} + (\varepsilon + \delta(-1)^n) u_n + \nu_n(t) \tag{93}$$

$$\nu_n' = \nu_{n+2} + h \nu_{n-2}. \tag{94}$$

**Remark.** It follows from this theorem that the equation

$$u_{n,t} = \frac{1}{u_{n-1}}\left(1 - \frac{u_n}{u_{n-2}}\right)$$

also has the symmetry (91), while the equation $u_{n,t} = u_{n-2}$ has the symmetry (93) with $\nu_n' = \nu_{n-2}$.

## 4. Applications of the transformations to the kink solutions for the Burgers equation

In the class of equations obtained in theorem 1, associated with the integrable non-point symmetries presented in section 3, is included, when $h(t) = 0$, the higher-order Burgers equation (4). This equation [2] possesses a trivial linear representation, but nevertheless has many of the properties of integrable nonlinear systems. In particular, it has an infinite class of exact solutions of kink type. The simplest kink solution is given by

$$u_n(t) = \frac{e^{-(n+1)k+e^{-2k}t} + e^{(n+1)k+e^{2k}t+n_0}}{e^{-nk+e^{-2k}t} + e^{nk+e^{2k}t+n_0}} \tag{95}$$

where $k$ and $n_0$ are two arbitrary parameters. Formula (95) is plotted in figure 1. Theorem 1 gives us the non-point integrable symmetry which can be associated with it

$$u_{n,\tau} = p_n u_n u_{n+1} - p_{n+1} u_n^2 + \delta(-1)^n u_n \tag{96}$$

with $p_n$ given by equation (9). Integrating this symmetry we obtain a non-point 1PG of transformations given by

$$\tilde{u}_n(t) = u_n + p_n u_n(\delta - u_{n+1})(e^{-\delta\tau} - 1) - p_{n+1}\frac{u_n(\delta - u_n)(e^{-\delta\tau} - 1)}{1 + (e^{-\delta\tau} - 1)(\delta - u_n)}. \tag{97}$$

In figure 2 we present the one-kink solution of figure 1 transformed under the transformation (97). This transformation acts differently on the even and odd points of the lattice. Moreover, the transformation changes when $\delta - u_n(t)$ changes sign. As one can see from figure 1, the initial kink solution, for the parameters chosen, has amplitude equal to $u_0(0) = 1$ at the origin $n = 0$. So the behaviour of the transformation is different if $\delta$ is less than or equal to 1 or is much greater than 1. In figure 2 one can find a plot of the one-kink solution (95) transformed by (97) with $\delta = 1$. In this case we can clearly see a different behaviour between even and odd points of the lattice and the difference in results when $u_n(t)$ is less than $\delta$ at the left of $n = 0$, where $u_{2k+1}(t) > u_{2k}(t)$, and when $u_n(t)$ is greater than $\delta$ at the right, where $u_{2k}(t) > u_{2k+1}(t)$. A different behaviour can be observed in figure 3 when

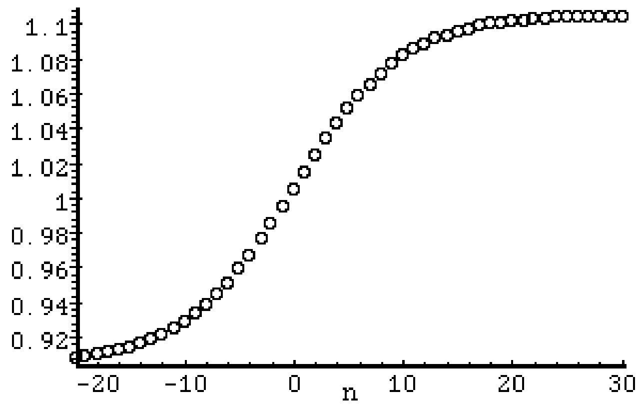


**Figure 1.** The one-kink solution of the differential-difference Burgers equation $u_n(t)$ for $k = 0.1$, $n_0 = 0$, $t = 0$.

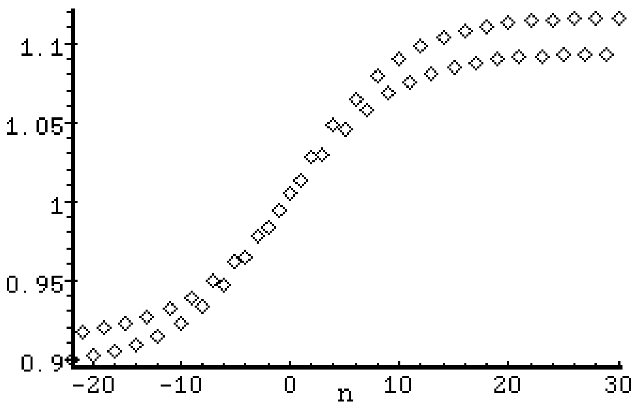


**Figure 2.** The transformed one-kink solution of the differential-difference Burgers equation $\tilde{u}_n(t)$ for $k = 0.1$, $n_0 = 0$, $t = 0$, $\tau = -0.095$ and $\delta = 1$.

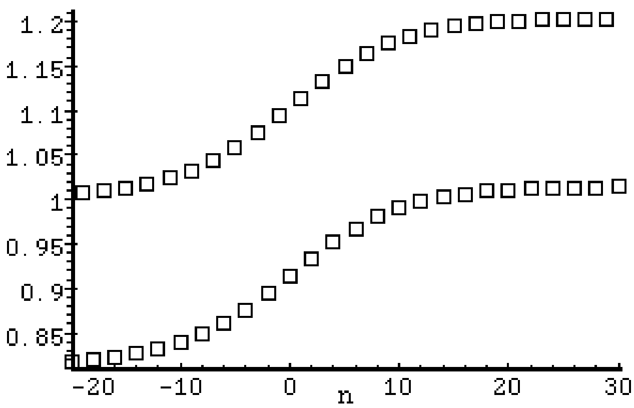


**Figure 3.** The transformed one-kink solution of the differential-difference Burgers equation $\tilde{u}_n(t)$ for $k = 0.1$, $n_0 = 0$, $t = 0$, $\tau = -0.091$ and $\delta = 2$.

$\delta$ is always greater than $u_n(t)$. In such a situation $u_n(t)$ is increased at the odd points of the lattice, while it is decreased at the even points of the lattice.

A similar behaviour can be observed for all the multikink solutions of the Burgers equation.

## 5. Conclusions

In this paper we have shown that in the case of differential difference equations we can extend the class of integrable symmetries from the case of intrinsic point symmetries [3], where the infinitesimal generator depends on the dependent field $u_n(t)$ only in the point $n$ of the lattice, to the case when the infinitesimal generator depends on a few neighbouring points of the lattice. To be able to integrate the symmetry generator and obtain a transformation which satisfies the conditions necessary to form a Lie group, we need the coefficients to depend explicitly on $n$. The transformations so obtained are such that they have no counterpart in the continuous limit when the lattice spacing goes to zero and the number of lattice points to infinity. Restricting ourselves, for the sake of simplicity, to the case of transformations depending on only two points in the lattice, $u_n(t)$ and $u_{n+1}(t)$, we find that the class of equations which possess such symmetries is given fundamentally by the differential-difference Burgers equation and the equations equivalent to it by point transformations. If we now start from the Burgers equation (4) and look for its integrable symmetries, it is easy to see that the following symmetries are part of its symmetry algebra:

$$u_{n,\tau_1} = p_n \left(t u_n u_{n+1} + \tfrac{1}{2}(n+1)\right) - p_{n+1}\left(t u_n^2 + \frac{n}{2}\frac{u_n}{u_{n-1}}\right) \tag{98}$$

$$u_{n,\tau_2} = p_n u_n u_{n+1} - p_{n+1} u_n^2 \qquad u_{n,\tau_3} = p_n - p_{n+1}\frac{u_n}{u_{n-1}} \tag{99}$$

$$u_{n,\tau_4} = (p_n - p_{n+1}) u_n. \tag{100}$$

These symmetries contain those presented previously in equation (96).

We can extend the class of symmetries, by considering group transformations of the form

$$\tilde{u}_n(t) = H_n(\tau, t, u_{n+N}(t), u_{n+N-1}(t), \ldots, u_{n-M}(t)) \qquad H_n(\tau = 0) = u_n(t). \tag{101}$$

Introducing the infinitesimal generator corresponding to (101)

$$h_n(t, u_{n+N}(t), u_{n+N-1}(t), \ldots, u_{n-M}(t)) = \frac{\mathrm{d}H_n}{\mathrm{d}\tau}(\tau = 0) \tag{102}$$

we can introduce the following differentiation operator:

$$D_\tau = \sum_{i=-\infty}^{\infty} h_{n+i} \frac{\partial}{\partial u_{n+i}}. \tag{103}$$

Consequently, the group closure condition becomes

$$H_n = \mathrm{e}^{\tau D_\tau} u_n \tag{104}$$

which, introducing the functions

$$\begin{aligned} &h_n^k(t, u_{n+N}(t), u_{n+N-1}(t), \ldots, u_{n-M}(t)) = \frac{\partial^k H_n}{\partial \tau^k}(\tau = 0) \\ &h_n^0 = u_n(t) \qquad h_n^1 = h_n \end{aligned} \tag{105}$$

is equivalent to the following conditions:

$$h_n^2 = D_\tau h_n \qquad h_n^3 = D_\tau h_n^2 \quad \ldots. \tag{106}$$

These conditions mean that the functions $h_n^k$ must depend only on the variables $u_{n+N}, \ldots, u_{n-M}$. In particular, in the case when $N = 1$ and $M = 1$, a function $H_n$ which satisfies the conditions (106) is given by

$$H_n = \lambda_n \mu_n A_n + \lambda_n (1 - \mu_n) B_n + (1 - \lambda_n)\mu_n C_n + (1 - \lambda_n)(1 - \mu_n) D_n \tag{107}$$

where

$$\begin{aligned} A_n &= A_n(\tau, t, u_{n-1}(t), u_n(t), u_{n+1}(t)) \\ B_n &= B_n(\tau, t, u_n(t), u_{n+1}(t)) \\ C_n &= C_n(\tau, t, u_{n-1}(t), u_n(t)) \end{aligned}$$

and

$$D_n = D_n(\tau, t, u_n(t))$$

with

$$\frac{\partial A_n}{\partial u_{n+1}} \frac{\partial A_n}{\partial u_{n-1}} \frac{\partial B_n}{\partial u_{n+1}} \frac{\partial C_n}{\partial u_{n-1}} \neq 0 \qquad \lambda_n \lambda_{n+1} = \mu_n \mu_{n+1} = 0 \qquad \text{for any } n.$$

Clearly, the same structure is also valid for the infinitesimal coefficient $h_n$.

We leave to a subsequent work the study of the equations associated with such higher non-point integrable symmetries together with a more detailed analysis of the triangular symmetries.

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